



Infinitely many weak solutions for a mixed boundary value system with (p_1, \dots, p_m) -Laplacian

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Abstract. The aim of this paper is to prove the existence of infinitely many weak solutions for a mixed boundary value system with (p_1, \dots, p_m) -Laplacian. The approach is based on variational methods.

Keywords: critical points, variational methods, infinitely many solutions, p -Laplacian.

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1 Introduction

The aim of this paper is to establish the existence of infinitely many weak solutions for the following mixed boundary value system with (p_1, \dots, p_m) -Laplacian.

$$\begin{cases} -(|u'_1|^{p_1-2}u'_1)' = \lambda F_{u_1}(t, u_1, \dots, u_m) & \text{in }]0, 1[\\ \vdots \\ -(|u'_m|^{p_m-2}u'_m)' = \lambda F_{u_m}(t, u_1, \dots, u_m) & \text{in }]0, 1[\\ u_i(0) = u'_i(1) = 0 & i = 1, \dots, m \end{cases} \quad (1.1)$$

where $m \geq 2$, $p_i > 1$ ($1 \leq i \leq m$), λ is a positive real parameter, $F: [0, 1] \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a C^1 -Carathéodory function such that $F(t, 0, \dots, 0) = 0$ for every $t \in [0, 1]$ and moreover we suppose that for every $\rho > 0$

$$\sup_{|(x_1, \dots, x_m)| \leq \rho} |F_{u_i}(t, x_1, \dots, x_m)| \in L^1([0, 1]), \quad i = 1, \dots, m.$$

Here F_{u_i} denotes the partial derivatives of F respect on u_i ($i = 1, \dots, m$).

Among the papers which have dealt with the nonlinear mixed boundary value problems we cite [1, 3, 10, 13].

We investigate the existence of infinitely many weak solutions for system (1.1) by using Theorem 1.1. This theorem is a refinement, due to Bonanno and Molica Bisci, of the variational principle of Ricceri [12, Theorem 2.5] and represents a smooth version of an infinitely many critical point theorem obtained in [5, Theorem 2.1].

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Theorem 1.1. *Let X be a reflexive Banach space, $\Phi: X \rightarrow \mathbb{R}$ is a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional, $\Psi: X \rightarrow \mathbb{R}$ is sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional, λ is a positive real parameter.*

Put, for each $r > \inf_X \Phi$

$$\begin{aligned} \varphi(r) &:= \inf_{u \in \Phi^{-1}(]-\infty, r])} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r])} \Psi(v) - \Psi(u)}{r - \Phi(u)}, \\ \gamma &:= \liminf_{r \rightarrow +\infty} \varphi(r), \quad \delta := \liminf_{r \rightarrow (\inf_X \Phi)^+} \varphi(r). \end{aligned} \tag{1.2}$$

One has

- (a) *For every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $\Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r])$ admits a global minimum, which is a critical point (local minimum) of $\Phi - \lambda\Psi$ in X .*
- (b) *If $\gamma < \infty$, then for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternatives hold: either*
 - (b₁) *$\Phi - \lambda\Psi$ possesses a global minimum, or*
 - (b₂) *there is a sequence $\{u_n\}$ of critical points (local minima) of $\Phi - \lambda\Psi$ such that $\lim_{n \rightarrow +\infty} \Phi(u_n) = +\infty$.*
- (c) *If $\delta < +\infty$, then for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternatives hold: either*
 - (c₁) *there is a global minimum of Ψ which is a local minimum of $\Phi - \lambda\Psi$, or*
 - (c₂) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of $\Phi - \lambda\Psi$, with $\lim_{n \rightarrow +\infty} \Phi(u_n) = \inf_X \Phi$ which weakly converges to a global minimum of Φ .*

Many authors proved the existence of infinitely many solutions by using the theorem above for different problems see for example [2, 4–9, 11].

The paper is arranged as follows. At first we prove the existence of an unbounded sequence of weak solutions of system (1.1) under some hypotheses on the behaviour of potential F at infinity (see Theorem 3.1). And as a consequence, we obtain the existence of infinitely many weak solutions for autonomous case (see Corollary 3.4).

2 Preliminaries

Let us introduce notation that will be used in the paper. Let

$$X_p = \{u \in W^{1,p}([0, 1]), \quad u(0) = 0\}, \quad p \geq 1$$

be the Sobolev space with the norm defined by

$$\|u\|_p = \left(\int_0^1 |u'(t)|^p dt \right)^{\frac{1}{p}}$$

for every $u \in X_p$, that is equivalent to the usual one.

It is well known that $(X_p, \|\cdot\|_p)$ is compactly embedded in $(C^0([0, 1]), \|\cdot\|_\infty)$ and one has

$$\|u\|_\infty \leq \|u\|_p \quad \forall u \in X_p. \tag{2.1}$$

Now, let X be the Cartesian product of m Sobolev spaces X_{p_i} , i.e. $X = \prod_{i=1}^m X_{p_i}$ endowed with the norm

$$\|u\| := \sum_{i=1}^m \|u_i\|_{p_i}$$

for all $u = (u_1, \dots, u_m) \in X$.

A function $u = (u_1, \dots, u_m) \in X$ is said a weak solution to system (1.1) if

$$\int_0^1 \sum_{i=1}^m |u'_i(t)|^{p_i-2} u'_i(t) v'_i(t) dt = \lambda \int_0^1 \sum_{i=1}^m F_{u_i}(t, u_1(t), \dots, u_m(t)) v_i(t) dt$$

for every $v = (v_1, \dots, v_m) \in X$.

In order to study system (1.1), we will use the functionals $\Phi, \Psi: X \rightarrow \mathbb{R}$ defined by putting

$$\Phi(u) := \sum_{i=1}^m \frac{\|u_i\|_{p_i}^{p_i}}{p_i}, \quad \Psi(u) := \int_0^1 F(t, u_1(t), \dots, u_m(t)) dt \quad (2.2)$$

for every $u = (u_1, \dots, u_m) \in X$.

Clearly, Φ is coercive, weakly sequentially lower semicontinuous and continuously Gâteaux differentiable and the Gâteaux derivative at point $u = (u_1, \dots, u_m) \in X$ is defined by

$$\Phi'(u)(v) = \int_0^1 \sum_{i=1}^m |u'_i(t)|^{p_i-2} u'_i(t) v'_i(t) dt$$

for every $v = (v_1, \dots, v_m) \in X$. On the other hand Ψ is well defined, weakly upper sequentially semicontinuous, continuously Gâteaux differentiable and the Gâteaux derivative at point $u = (u_1, \dots, u_m) \in X$ is defined by

$$\Psi'(u)(v) = \int_0^1 \sum_{i=1}^m F_{u_i}(x, u_1(t), \dots, u_m(t)) v_i(t) dt$$

for every $v = (v_1, \dots, v_m) \in X$.

A critical point for the functional $I_\lambda := \Phi - \lambda\Psi$ is any $u \in X$ such that

$$\Phi'(u)(v) - \lambda\Psi'(u)(v) = 0 \quad \forall v \in X.$$

Hence, the critical points for functional $I_\lambda := \Phi - \lambda\Psi$ are exactly the weak solutions to system (1.1).

A function $u: [0, 1] \rightarrow \mathbb{R}^m$ is said a solution to system (1.1) if $u \in C^1([0, 1], \mathbb{R}^m)$, $|u'_i|^{p_i-2} u'_i$ is AC($[0, 1]$) ($i = 1, \dots, m$) and the system (1.1) is satisfied a.e.

Standard methods show that solutions to system (1.1) coincide with weak ones when F is a C^1 function.

Now, put

$$A = \liminf_{r \rightarrow +\infty} \frac{\int_0^1 \max_{\xi \in Q(r)} F(t, \xi_1, \dots, \xi_m) dt}{r^s}, \quad (2.3)$$

where $s = \min_{1 \leq i \leq m} \{p_i\}$, $Q(r) = \{\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m : \sum_{i=1}^m |\xi_i| \leq r\}$.

$$B = \limsup_{|\xi| \rightarrow +\infty, \xi \in \mathbb{R}_+^m} \frac{\int_0^1 F(t, \xi_1, \dots, \xi_m) dt}{\sum_{i=1}^m |\xi_i|^{p_i}}, \quad (2.4)$$

$$\lambda_1 = \frac{1}{B}, \quad \lambda_2 = \frac{1}{\left(\sum_{i=1}^m p_i^{\frac{1}{p_i}}\right)^s A}, \quad (2.5)$$

we suppose $\lambda_1 = 0$ if $B = +\infty$, and $\lambda_2 = +\infty$ if $A = 0$,

$$\bar{k} = \max_{1 \leq i \leq m} \left\{ \frac{2^{p_i-1}}{p_i} \right\}. \quad (2.6)$$

3 Main results

Our main result is the following theorem.

Theorem 3.1. *Assume that*

(i₁) $F(t, x) \geq 0$ for every $(t, x) \in [0, 1] \times \mathbb{R}_+^m$, where $\mathbb{R}_+^m = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : x_i \geq 0, i = 1, \dots, m\}$;

(i₂)

$$\begin{aligned} & \liminf_{r \rightarrow +\infty} \frac{\int_0^1 \max_{\xi \in Q(r)} F(t, \xi_1, \dots, \xi_m) dt}{r^s} \\ & < \frac{1}{\left(\sum_{i=1}^m p_i^{\frac{1}{p_i}}\right)^s} \limsup_{|\xi| \rightarrow +\infty, \xi \in \mathbb{R}_+^m} \frac{\int_{\frac{1}{2}}^1 F(t, \xi_1, \dots, \xi_m) dt}{\sum_{i=1}^m |\xi_i|^{p_i}}, \end{aligned}$$

where $Q(r) = \{\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m : \sum_{i=1}^m |\xi_i| \leq r\}$ and $s = \min_{1 \leq i \leq m} \{p_i\}$.

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, where λ_1, λ_2 are given by (2.5), the system (1.1) has a sequence of weak solutions which is unbounded in X .

Proof. Our goal is to apply Theorem 1.1 (b). Consider the Sobolev space X and the operators defined in (2.2). Pick $\lambda \in]\lambda_1, \lambda_2[$.

Let $\{c_n\}$ be a real sequence such that $\lim_{n \rightarrow +\infty} c_n = +\infty$ and

$$\lim_{n \rightarrow +\infty} \frac{\int_0^1 \max_{\xi \in Q(c_n)} F(t, \xi_1, \dots, \xi_m) dt}{c_n^s} = A.$$

Put

$$r_n = \frac{c_n^s}{\left(\sum_{i=1}^m p_i^{\frac{1}{p_i}}\right)^s}$$

for all $n \in \mathbb{N}$.

Taking into account (2.1), one has $\sum_{i=1}^m |v_i(t)| < c_n$ where $v = (v_1, \dots, v_m) \in X$ such that $\sum_{i=1}^m \frac{\|v_i\|_{p_i}^{p_i}}{p_i} < r_n$.

Hence, for all $n \in \mathbb{N}$, one has

$$\begin{aligned} \varphi(r_n) &= \inf_{(u_1, \dots, u_m) \in \Phi^{-1}([-\infty, r_n])} \frac{\sup_{(v_1, \dots, v_m) \in \Phi^{-1}([-\infty, r_n])} \Psi(v_1, \dots, v_m) - \Psi(u_1, \dots, u_m)}{r_n - \Phi(u_1, \dots, u_m)} \\ &\leq \frac{\sup_{(v_1, \dots, v_m) \in \Phi^{-1}([-\infty, r_n])} \int_0^1 F(t, v_1(t), \dots, v_m(t)) dt}{r_n} \\ &\leq \left(\sum_{i=1}^m p_i^{\frac{1}{p_i}} \right)^s \frac{\int_0^1 \max_{\xi \in Q(c_n)} F(t, \xi_1, \dots, \xi_m) dt}{c_n^s}, \end{aligned}$$

therefore, since from (i_2) one has $A < \infty$, we obtain

$$\gamma := \liminf_{n \rightarrow \infty} \varphi(r_n) \leq \left(\sum_{i=1}^m p_i^{\frac{1}{p_i}} \right)^s A < \infty.$$

Now, fix $\lambda \in]\lambda_1, \lambda_2[$, we claim that the functional $I_\lambda = \Phi - \lambda\Psi$ is unbounded from below. Let $\{\zeta_n = (\zeta_{in})_{i=1, \dots, m}\}$ be a real sequence such that $\lim_{n \rightarrow \infty} |\zeta_n| = +\infty$ and

$$\lim_{n \rightarrow +\infty} \frac{\int_{\frac{1}{2}}^1 F(t, \zeta_{1n}, \dots, \zeta_{mn}) dt}{\sum_{i=1}^m |\zeta_{in}|^{p_i}} = B. \quad (3.1)$$

For all $n \in \mathbb{N}$ define

$$\omega_{in}(t) = \begin{cases} 2\zeta_{in}t & \text{if } t \in [0, \frac{1}{2}[\\ \zeta_{in} & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \quad i = 1, \dots, m$$

clearly, $\omega_n = (\omega_{1n}, \dots, \omega_{mn}) \in X$ and

$$\Phi(\omega_n) = \sum_{i=1}^m \frac{1}{p_i} \|\omega_{in}\|_{p_i}^{p_i} \leq \bar{k} \sum_{i=1}^m |\zeta_{in}|^{p_i} \quad (3.2)$$

where \bar{k} is given by (2.6).

Taking into account (i_1) , we have

$$\int_0^1 F(t, \omega_n(t)) dt \geq \int_{\frac{1}{2}}^1 F(t, \zeta_{1n}, \dots, \zeta_{mn}) dt. \quad (3.3)$$

Then, by using (3.2) and (3.3) for all $n \in \mathbb{N}$ we have

$$\Phi(\omega_n) - \lambda\Psi(\omega_n) \leq \bar{k} \sum_{i=1}^m |\zeta_{in}|^{p_i} - \lambda \int_{\frac{1}{2}}^1 F(t, \zeta_{1n}, \dots, \zeta_{mn}) dt. \quad (3.4)$$

Now, if $B < \infty$, we fix $\epsilon \in]\frac{\bar{k}}{\lambda B}, 1[$, from (3.1) there exists $\nu_\epsilon \in \mathbb{N}$ such that

$$\int_{\frac{1}{2}}^1 F(t, \zeta_{1n}, \dots, \zeta_{mn}) dt > \epsilon B \sum_{i=1}^m |\zeta_{in}|^{p_i} \quad \forall n > \nu_\epsilon$$

therefore

$$\Phi(\omega_n) - \lambda\Psi(\omega_n) \leq \left(\bar{k} - \lambda \epsilon B \right) \sum_{i=1}^m |\zeta_{in}|^{p_i} \quad \forall n > \nu_\epsilon$$

by the choice of ϵ , one has

$$\lim_{n \rightarrow \infty} (\Phi(\omega_n) - \lambda \Psi(\omega_n)) = -\infty.$$

On the other hand, if $B = +\infty$, we fix

$$M > \frac{\bar{k}}{\lambda}$$

from (3.1) there exists $\nu_M \in \mathbb{N}$ such that

$$\int_{\frac{1}{2}}^1 F(t, \xi_{1n}, \dots, \xi_{mn}) dt > M \sum_{i=1}^m |\xi_{in}|^{p_i} \quad \forall n > \nu_M$$

therefore

$$\Phi(\omega_n) - \lambda \Psi(\omega_n) \leq (\bar{k} - \lambda M) \sum_{i=1}^m |\xi_{in}|^{p_i} \quad \forall n > \nu_M$$

by the choice of M , one has

$$\lim_{n \rightarrow \infty} (\Phi(\omega_n) - \lambda \Psi(\omega_n)) = -\infty.$$

Hence, our claim is proved.

Since all assumptions of Theorem 1.1 (b) are verified, the functional $I_\lambda = \Phi - \lambda \Psi$ admits a sequence $\{u_n\}$ of critical points such that $\lim_{n \rightarrow \infty} \|u_n\| = +\infty$ and the conclusion is achieved. \square

Remark 3.2. In Theorem 3.1 we can replace $r \rightarrow +\infty$ by $r \rightarrow 0^+$, applying in the proof part (c) of Theorem 1.1 instead of (b). In this case a sequence of pairwise distinct weak solutions to the system (1.1) which converges uniformly to zero is obtained.

Remark 3.3. We consider the system

$$\begin{cases} -(|u'_1|^{p_1-2}u'_1)' + |u_1|^{p_1-2}u_1 = \lambda F_{u_1}(t, u_1, \dots, u_m) & \text{in }]0, 1[\\ \vdots \\ -(|u'_m|^{p_m-2}u'_m)' + |u_m|^{p_m-2}u_m = \lambda F_{u_m}(t, u_1, \dots, u_m) & \text{in }]0, 1[\\ u_i(0) = u'_i(1) = 0 & i = 1, \dots, m \end{cases} \quad (3.5)$$

by using the usual norm

$$\|u\|_{p_i} = \left(\int_0^1 |u(t)|^{p_i} dt + \int_0^1 |u'(t)|^{p_i} dt \right)^{\frac{1}{p_i}}$$

in X_{p_i} , and the constant $\bar{k} = \max_{1 \leq i \leq m} \left\{ \frac{2+p_i+2p_i(p_i+1)}{2p_i(p_i+1)} \right\}$ we can prove in a very similar way to that used to prove Theorem 3.1, that for each $\lambda \in]\lambda_1, \lambda_2[$, with λ_1 and λ_2 given by (2.5), the system (3.5) has a sequence of weak solutions which is unbounded in X .

Now, we point out a special case of Theorem 3.1.

Corollary 3.4. Let $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$ be two positive continuous functions such that the differential 1-form $\omega = f(x, y) dx + g(x, y) dy$ is integrable and let F be a primitive of ω with $F(0, 0) = 0$. Fix $p, q > 1$ with $p \leq q$ assume that

$$\liminf_{r \rightarrow +\infty} \frac{F(r, r)}{r^p} = 0, \quad \limsup_{r \rightarrow +\infty} \frac{F(r, r)}{r^q} = +\infty.$$

Then, the system

$$\begin{cases} -(|u'|^{p-2}u')' = f(u, v) & \text{in } I =]0, 1[\\ -(|v'|^{q-2}v')' = g(u, v) & \text{in } I =]0, 1[\\ u(0) = u'(1) = 0 \\ v(0) = v'(1) = 0 \end{cases}$$

possesses a sequence of pairwise distinct solutions which is unbounded in X .

Proof. Since f and g are positive one has that $\max_{(\xi, \eta) \in Q(r)} F(\xi, \eta) \leq F(r, r)$ for every $r \in \mathbb{R}_+$.

Therefore

$$\liminf_{r \rightarrow +\infty} \frac{\int_0^1 \max_{(\xi, \eta) \in Q(r)} F(\xi, \eta) dt}{r^p} \leq \liminf_{r \rightarrow +\infty} \frac{F(r, r)}{r^p} = 0$$

on the other hand, we have

$$+\infty = \frac{1}{2} \limsup_{r \rightarrow +\infty} \frac{F(r, r)}{r^q} \leq \limsup_{r \rightarrow +\infty} \frac{F(r, r)}{r^p + r^q} \leq \limsup_{\sqrt{\xi^2 + \eta^2} \rightarrow +\infty, (\xi, \eta) \in \mathbb{R}_+^2} \frac{F(\xi, \eta)}{\xi^p + \eta^q},$$

then we have $\lambda_1 = 0$ and $\lambda_2 = +\infty$ and all assumptions of Theorem 3.1 are satisfied and the proof is complete. \square

Now, we present one example that illustrates our result.

Example 3.5. Consider $p = q = 4$ and the function $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$F(x, y) = \begin{cases} x^2 y^2 e^{2(\sin \log x + 1)} e^{2(\sin \log y + 1)} & \text{if } x > 0, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $f(x, y)$ and $g(x, y)$ the partial derivatives of F respect on x and y respectively

$$f(x, y) = \begin{cases} 2xy^2 e^{2(\sin \log x + 1)} e^{2(\sin \log y + 1)} [1 + \cos \log x] & \text{if } x > 0, y > 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$g(x, y) = \begin{cases} 2x^2 y e^{2(\sin \log x + 1)} e^{2(\sin \log y + 1)} [1 + \cos \log y] & \text{if } x > 0, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since f and g are non negative one has that $\max_{(x, y) \in Q(r)} F(x, y) \leq F(r, r)$ for every $r \in \mathbb{R}_+$. By a simple computation, we obtain

$$\liminf_{r \rightarrow +\infty} \frac{\max_{(x, y) \in Q(r)} F(x, y)}{r^4} \leq \liminf_{r \rightarrow +\infty} \frac{F(r, r)}{r^4} = 1$$

$$\limsup_{\sqrt{x^2 + y^2} \rightarrow +\infty, (x, y) \in \mathbb{R}_+^2} \frac{F(x, y)}{2(x^4 + y^4)} = \frac{e^8}{4}.$$

Hence, from Theorem 3.1, for each $\lambda \in]\frac{4}{e^8}, \frac{1}{2^6}[$ the system

$$\begin{cases} -(|u'|^2 u')' = \lambda f(u, v) & \text{in } I =]0, 1[\\ -(|v'|^2 v')' = \lambda g(u, v) & \text{in } I =]0, 1[\\ u(0) = u'(1) = 0 \\ v(0) = v'(1) = 0 \end{cases}$$

has a sequence of solutions which is unbounded in $X = X_4 \times X_4$.

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